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Routings for involutions of a hypercube

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Abstract

Given a pairing of the vertices of a hypercube, we study the existence of a set of paths between paired vertices, such that no edge is assigned to more than one path. For a multiprocessor whose processors are interconnected in a hypercube topology, such a set of paths could be used for circuit switching of messages between paired processors (vertices); each path would be a 2-way communication link. We resolve completely the existence of such a set of paths, where the pairing forms an automorphism of the hypercube. In particular, if the dimension d of the hypercube is odd, such a set of paths always exists. If d is even, exactly two classes of involutory automorphism (the antipodal automorphism and one other "nearly antipodal" class) lack such paths.

1. Introduction

The d-dimensional *hypercube* (or *d-cube*) H_d is the graph having all d-tuples over $\{0,1\}$ as vertex set, and two vertices adjacent if they differ in exactly one coordinate. For any pair of vertices of a hypercube, the number of edges in a shortest path between them is called the *distance* between them and written dist(.,.). This distance equals the number of coordinates in which the two vertices differ, so it equals the usual Hamming distance. For all vertices v, w in H_d, $\mathrm{dist}(v, w) \le d$. For each vertex v of H_d,

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there is exactly one vertex w such that $\mathrm{dist}(v, w) = d$; w is called the *antipode* of v. When $v = x_1 x_2 \ldots x_d$, the antipode of v is $\bar{x}_1 \bar{x}_2 \ldots \bar{x}_d$. (Here, and throughout the paper, $\bar{0} = 1$ and $\bar{1} = 0$.)

All graphs we deal with are undirected. We will say that a *pairing* on a set S is a permutation π on S, so that π^2 is the identity map. An *automorphism* of a graph $G = (V, E)$ is a permutation π on V, such that for all $v, w \in V$, $\pi(v)$ is adjacent to $\pi(w)$ iff v is adjacent to w. An *involution* is an automorphic pairing on the vertices of G. Equivalently, π is an involution if π is an automorphism and for all $v, w \in V$, $w = \pi(v)$ iff $v = \pi(w)$. The *antipodal* automorphism of H_d is the involution π which maps each vector $x_1 x_2 \ldots x_d$ to its antipode $\bar{x}_1 \bar{x}_2 \ldots \bar{x}_d$.

Two permutations π_1 and π_2 on the set of vertices of a graph G are said to be *equivalent* if there exists an automorphism ρ of G such that $\pi_2 = \rho^{-1} \pi_1 \rho$.

Let π be a pairing on the set of vertices of H_d, and P a set of paths in H_d. P is said to *route* π (or to be a *routing* of π) if:

(1) There is a one-to-one correspondence between paths in P and (unordered) pairs v, $\pi(v)$ of π; under this correspondence, for each pair v, $\pi(v)$, the initial and final vertices of the corresponding path are v and $\pi(v)$ (or $\pi(v)$ and v).

(2) The paths in P are edge disjoint: no edge is in more than one path in P.

Where $v = \pi(v)$, in practice the corresponding path in P has length 0, and is ignored. Figs. 1 and 2 display routings for certain pairings in H_3 and H_4. A pairing is said to be *routable* if it has a routing.

The theorem we prove states that an involution π of H_d is routable unless d is even and π is antipodal or nearly antipodal. (The inequality on d in (2)(ii) is essential, as is shown by Example 1 in Section 3.)

Theorem. *Let π be an involution of H_d. Then:*

(1) *If d is odd, then π is routable.*

(2) *If d is even, π is routable iff neither of the following two conditions applies:*

(i) *π is the antipodal involution, with $d \geq 2$,*

(ii) *π is equivalent to an involution $\pi'(x_1 x_2 \ldots x_d) = y_1 y_2 \ldots y_d$ where $y_i = \bar{x}_i$ for $i \leq d-2$, $y_{d-1} = x_d$, and $y_d = x_{d-1}$, with $d \geq 4$.*

Several multiprocessor computers are designed such that the processors and the network connecting the processors are configured as a hypercube. Each processor has its own local memory; there is no shared memory. Instead, processors communicate by message passing. Communication links of the hypercube (as a multicomputer) correspond to edges of the hypercube (as a graph): messages are constrained to travel along edges of the graph. In the mode of communication called circuit switching, a path is assigned to each sender/receiver pair to serve as a virtual communication channel through which a sequence of messages can be transferred [3]. Szymanski studied circuit switching of a hypercube [17]. He was concerned with establishing, by means of edge-disjoint directed paths, a 1-way connection specified by an arbitrary permutation as the source-destination mapping. As is natural for his purpose, he modeled a hypercube as a directed graph with two complementary edges between each pair of adjacent vertices. He conjectured that such a directed routing exists for every permutation of vertices of a d-cube, and showed that this is so for $d \leq 3$.

There are two approaches to circuit switching for 2-way connection between paired processors. One is to consider the problem as a special case of the problem Szymanski studied, where the source-destination mapping is restricted to permutations which are pairings. The other way is to model, as we do here, a hypercube as an undirected graph and route each pair through an undirected path. These are two different problems with different practical implications: in the first approach, two directed paths connecting a pair of vertices can traverse different sets of vertices, a flexibility for solutions but a possible complication for implementations. The methods used in this paper can also be applied to the directed path approach; the proof that a directed routing exists for every involution of a hypercube is much easier than the proof here concerning existence and nonexistence of undirected routings.

Efficient methods for performing various patterns of communication of a hypercube are studied in [5, 15, 16]; in particular, [16] includes arbitrary permutations but uses packet switching, so edge-disjoint paths are not employed. A different application of edge-disjoint undirected paths in a hypercube is given in [1], where such paths are used in establishing virtual subcubes in a multiuser hypercube system (as an alternative to job migration).

Very little seems to be known about routings for arbitrary pairings in a hypercube. The related problem of vertex-disjoint paths is treated in [8]; there it is shown that for any $2k$ distinct vertices $s_1,t_1,s_2,t_1,\ldots,s_k,t_k$ of H_d, where $k \leq (d+1)/2$ and $(d,k) \neq (3,2)$, there exist k vertex-disjoint paths $p_1,p_1,\ldots,p_k$ such that each p_i is a path from s_i to t_i.

The problem we treat, edge-disjoint paths on a hypercube, is a special case of the problem of finding a multicommodity flow in an undirected graph. Multicommodity flows in general graphs are not well understood; however this topic is better developed for planar graphs in which all terminals are on the same face [2, 12]. Some work on edge-disjoint paths, normally in meshes, has been motivated by the routings of wires in VLSI circuits [7, 11, 13]. The result of this paper suggests the difficulties involved in a general theory of edge-disjoint paths in undirected graphs.

2. Involutions

It is clear that if π_1 and π_2 are equivalent involutions of H_d, then π_1 is routable iff π_2 is. (The automorphism ρ relating π_1 and π_2 maps any routing of π_2 to a routing of π_1, and ρ^{-1} performs the reverse map.) Thus to prove the theorem, it is sufficient to consider one involution of each equivalence class. In this section we classify involutions of a hypercube.

We start with a lemma which says, roughly, that every involution of a hypercube may be obtained by (1) permuting coordinates, and (2) at chosen coordinates, exchanging the values 0 and 1. For completeness, we include its proof.

Lemma. *Let π be an automorphism of hypercube H_d. Then there exist a permutation μ of coordinates and bijections $\tau_1, \tau_2, \ldots, \tau_d$ on $\{0, 1\}$ such that $\pi(x_1 x_2 \ldots x_d) = y_1 y_2 \ldots y_d$, where $y_{\mu(i)} = \tau_{\mu(i)}(x_i)$. Note: there are only two bijections on $\{0, 1\}$: $0 \to 1$, $1 \to 0$, and $0 \to 0$, $1 \to 1$.*

Proof. In this proof the *dimension i neighbor* of a vertex v is that neighbor of v which (as a tuple) differs from v in the ith coordinate. Suppose an automorphism π of H_d is given. Let v_0 be the vertex of H_d in which every coordinate is 0. Since π maps each neighbor of v_0 one-to-one to a neighbor of $\pi(v_0)$, there must be a permutation μ of coordinates so that π maps the dimension i neighbor of v_0 to the dimension $\mu(i)$ neighbor of $\pi(v_0)$. Choose bijections $\tau_1, \tau_2, \ldots, \tau_d$ on $\{0, 1\}$ so that τ_i is the identity if the ith coordinate of $\pi(v_0)$ is 0, and the exchange otherwise. Let π' be a bijection on H_d defined by $\pi'(x_1 x_2 \ldots x_d) = y_1 y_2 \ldots y_d$ where $y_{\mu(i)} = \tau_{\mu(i)}(x_i)$. We show that $\pi(v) = \pi'(v)$ hold for every vertex v of H_d, by induction on the distance of v from v_0.

First of all, note that π' is an automorphism of H_d, because if vertices u and w differ in (and only in) coordinate i, then $\pi'(u)$ and $\pi'(w)$ differ in (and only in) coordinate $\mu(i)$. For the basis of the induction, it is easy to verify that the choice of μ and τ_i $(1 \leq i \leq d)$ forces π' to agree with π on v_0 and on all of its neighbors. Now assume $\mathrm{dist}(v, v_0) \geq 2$. Then at least two coordinates of v, say the first and the second, must be 1. So let $v = 11x_3 \ldots x_d$. By the induction hypothesis, π and π' agree on $00x_3 \ldots x_d$, $01x_3 \ldots x_d$, and $10x_3 \ldots x_d$. Therefore, since π and π' are automorphisms, each of $\pi(v)$ and $\pi'(v)$ must be adjacent to both $\pi(01x_3 \ldots x_d)$ and $\pi(10x_3 \ldots x_d)$, and must be different from $\pi(00x_3 \ldots x_d)$. There is only one such vertex in H_d, hence $\pi(v) = \pi'(v)$. This completes the induction. $\square$

Let π be an automorphism of H_d, and μ and τ_i $(1 \leq i \leq d)$ be the associated mappings described in the Lemma. Since the permutation of coordinates associated with π^2 is μ^2, it follows that if π is an involution, then μ is an involution; further, if $\mu(i) = j$, then $\tau_i = \tau_j$. Consequently, given an involution of H_d, every coordinate i is of one of the following four types:

- Type-1 coordinate: μ fixes i, and τ_i is the identity map.
- Type-2 coordinate: μ fixes i, and τ_i exchanges 0 and 1.
- Type-3 coordinate: μ exchanges i with another coordinate j $(j \neq i)$, and $\tau_j = \tau_i$ is the identity map.
- Type-4 coordinate: μ exchanges i with another coordinate j $(j \neq i)$, and $\tau_j = \tau_i$ exchanges 0 and 1.

We say that the involution π of H_d is of *class* (c_1, c_2, c_3, c_4), where $\sum c_i = d$, if the number of type-i coordinates is c_i, for each i.

Clearly, two involutions of the same class are equivalent. (They are related by an easily constructed automorphism that permutes coordinates so that each coordinate is mapped to a coordinate of the same type, and each exchanging pair of type 3 or 4 is mapped to an exchanging pair of the same type.) Furthermore, if π is an involution and i is a type-4 coordinate then, by choosing an automorphism ρ such that

$\rho(x_1x_2 \ldots x_d) = y_1y_2 \ldots y_d$ where $y_j = x_j$ for all $j \neq i$ and $y_i = \bar{x}_i$, π is equivalent to $\rho^{-1}\pi\rho$, which has one fewer pair of type-4 coordinates. Hence every involution is equivalent to one having no type-4 coordinate. Since routability is preserved by equivalence, and every involution is equivalent to one having no type-4 coordinates, to prove the theorem it is sufficient to consider involutions with no type-4 coordinate.

3. Examples

The following examples illustrate the concepts explained above, and also serve as the fundamental constructions that the recursive constructions in Propositions 3–5 build upon.

Example 1. Let α be the involution on the 2-cube which maps each vertex x_1x_2 to x_2x_1. Then α is of class (0, 0, 2, 0). The set consisting of the single path (01, 00, 10) is a routing of α.

Example 2. Let β be the involution on the 3-cube which maps each vertex $x_1x_2x_3$ to $\bar{x}_1x_3x_2$. Then β is of class (0, 1, 2, 0).

Example 3. Let γ be the antipodal involution on the 3-cube. Then γ maps each vertex $x_1x_2x_3$ to $\bar{x}_1\bar{x}_2\bar{x}_3$, and is of class (0, 3, 0, 0). A routing of γ is displayed in Fig. 1(b).

Example 4. Let δ be the involution on the 3-cube which maps each vertex $x_1x_2x_3$ to $x_1\bar{x}_2\bar{x}_3$. Then δ is of class (1, 2, 0, 0). A routing of δ is displayed in Fig. 1(c).

Example 5. Let ω be the class (0, 0, 4, 0) involution defined by $\omega(x_1x_2x_3x_4) = x_2x_1x_4x_3$. A routing of ω is shown in Fig. 2.

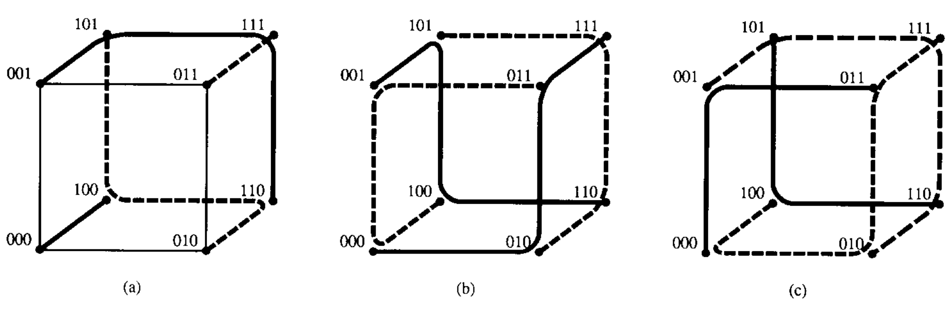

Fig. 1. Routings for three involutions. Paths of the routings are thick solid or dashed lines. Routing for involution β of Example 2 is shown in (a); routings for involutions γ and δ of Examples 3 and 4 are in (b) and (c) respectively.

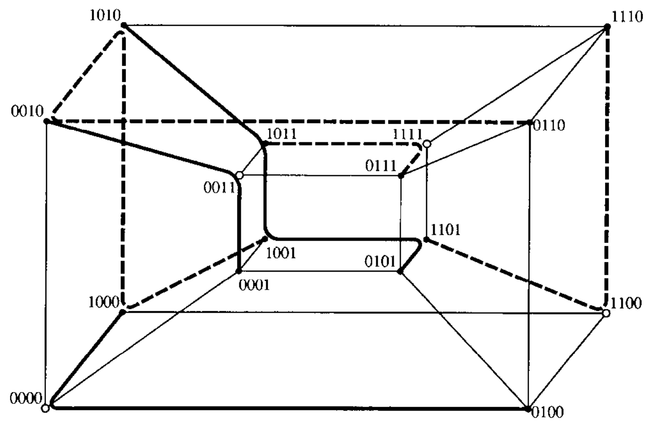

Fig. 2. Routing for the involution ω of Example 5. White circles indicate fixed points of the involution.

4. Antipodal and nearly antipodal involutions

In this section we show that certain involutions are not routable. In particular, an antipodal automorphism on a hypercube of even dimension d ($d \geq 2$) is not routable. Also, an involution of class $(0, d-2, 2, 0)$, where d is even and $d \geq 4$, is not routable. That every other involution (with no type-4 coordinate) is routable is shown in the next section.

Proposition 1. *Let π be an antipodal automorphism on the d-cube H_d ($d > 0$). If the set P of paths routes π then d is odd.*

Proof. In this proof we will show that (1) every edge of H_d appears in P, and (2) the degree of each vertex is odd. The latter is equivalent to d being odd.

Since H_d has 2^d vertices, none of which is fixed by π (since $d > 0$), P contains 2^{d-1} paths. Since the distance between v and $\pi(v)$ is d for each vertex v, each path contains at least d edges. Hence at least $d2^{d-1}$ distinct edges appear in P (since paths in P are edge disjoint). Since H_d has only $d2^{d-1}$ edges, then every edge of H_d appears in P.

Consider a vertex v_0. Exactly one path in P has v_0 as an endvertex; every other path contains an even number of edges incident with v_0. Then an odd number of edges incident with v_0 are in P. Since every edge incident with v_0 is in P, v_0 is incident with an odd number of edges in H_d. Since v_0 is incident with d edges in the d-cube H_d, d is odd. $\square$

Proposition 2. *Let π be a class $(0, d-2, 2, 0)$ involution on H_d, where d is even, and $d \geq 4$. Then π is not routable.*

(Note: For the case $d = 2$, Example 1 displays a routing for π.)

Proof. Let $d \geq 4$. Let π be defined by $\pi(x_1x_2 \dots x_d) = y_1y_2 \dots y_d$, where $y_1 = x_2$, $y_2 = x_1$, and $y_i = \bar{x}_i$ for $i > 2$. Let $v = x_1x_2 \dots x_d$. If $x_1 \neq x_2$ then $\operatorname{dist}(v, \pi(v)) = d$, while if $x_1 = x_2$ then $\operatorname{dist}(v, \pi(v)) = d - 2$. Then $\sum\{\operatorname{dist}(v, \pi(v))\colon v \in H_d\} = 2^d(d-1)$. Call this quantity α. α has been computed as a sum over all vertices of H_d. Then $\alpha/2$ is the corresponding sum over all unordered pairs of vertices (v, w) such that $w = \pi(v)$.

Suppose P is a routing for π. Then, by the definition of routing, $\alpha/2 = \sum\{\operatorname{dist}(p_0, p_f)\colon p$ is a path in P, p_0 its initial vertex, and p_f its final vertex$\}$. (It follows from this that P contains at least $\alpha/2$ edges.)

Since $d > 2$, at least one coordinate is complemented by π, so π fixes no vertex of H_d. Accordingly, each vertex v of H_d is an endvertex of exactly one path in P. Since one path in P contains one edge incident with v, and all other paths in P contain an even number of edges incident with v, P contains an odd number of edges which are incident with v. Then P contains at most $d - 1$ edges incident with v (since d is even). Then P contains at most $2^{d-1}(d-1)$ edges (which we note is equal to $\alpha/2$).

In the previous two paragraphs we established (1) $\alpha/2$ equals the sum of the distances between the initial and final vertices of paths in P, and (2) $\alpha/2$ is both an upper and lower bound on the number of edges in P. Hence P contains exactly $\alpha/2$ edges. More importantly, it follows that every path in P is a geodesic path. (A *geodesic path* from x to y is a shortest path from x to y: a path having $\operatorname{dist}(x, y)$ edges.)

Denote by H_{d-2} the set of vertices of H_d having first two coordinates both 0; H_{d-2} is a $(d-2)$-cube. For each vertex v of H_{d-2}, $\pi(v)$ is also in H_{d-2} (since π exchanges the first two coordinates). Thus the restriction of π to H_{d-2} (which we will call π_{d-2}) is an involution of H_{d-2}; in particular, π_{d-2} is the antipodal involution of H_{d-2}.

Let $v \in H_{d-2}$, and p be the path in P having v as initial vertex. Since p is a geodesic path from v to $\pi(v)$, every edge of p lies within H_{d-2}. (Indeed, if $p = (v, v_1, v_2, \dots, v_{d-2})$ where $v_{d-2} = \pi(v)$, and $v_i \in H_{d-2}$ while $v_{i+1} \notin H_{d-2}$, then v_{i+1} differs from $\pi(v)$ in more coordinates than v_i does.) Let P_{d-2} be the set of paths in P whose initial vertex is in H_{d-2}. Then P_{d-2} is a routing (in H_{d-2}) for π_{d-2}. However, since $d - 2 \geq 2$ and is even, the antipodal involution π_{d-2} is not routable. We have reached a contradiction; hence the assumption that π is routable is false. $\square$

5. Proof of the Theorem

In this section we construct routings for involutions which are routable. The constructions are recursive, and rely on the examples in Section 3.

Let H be a $(d-2)$-cube, and H_d be a d-cube (where $d \geq 2$). H_d contains four disjoint copies of H. Expressing this more formally, there are four embeddings $\phi_0, \phi_1, \phi_2, \phi_3$ of H into H_d; ϕ_0 maps each vertex $x_1x_2 \dots x_{d-2}$ of H to $x_1x_2 \dots x_{d-2}00$. Similarly, ϕ_1, ϕ_2, ϕ_3 map vertices of H to vertices of H_d whose last two coordinates are respectively 01, 10, and 11. In Fig. 3 these four disjoint subcubes are displayed as circles. We will occasionally write $\phi_0(H)$ as $H00$, $\phi_1(H)$ as $H01$, etc. We will call an edge of H_d *internal* if it joins two vertices in $\phi_i(H)$, for the same i; equivalently, an edge

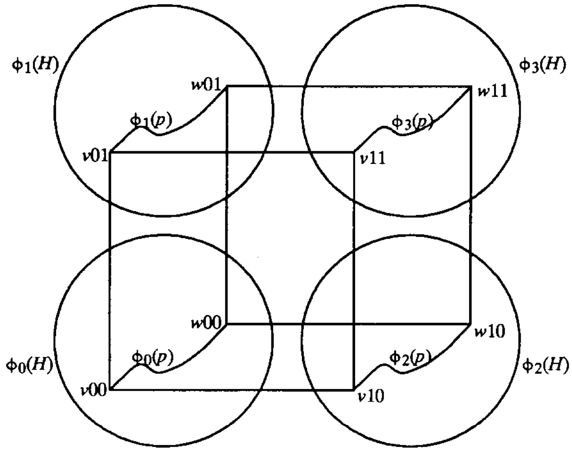

Fig. 3. Anatomy of a d-cube, showing vertices and edges of a virtual 3-cube $C(p)$ (where p is a path in the $(d-2)$-cube H having endpoints v and w).

of H_d is internal if its two endpoints differ in one of the first $d-2$ coordinates. An edge of H_d is *external* if it is not internal, that is, if its two endpoints differ in one of the last two coordinates.

We have special interest in certain squares and "virtual 3-cubes" in H. For a vertex v of H, let $S(v)$ denote the set $\{\phi_i(v)\colon 0 \le i \le 3\}$ of four vertices of H_d, together with the four edges of H_d joining them (namely, $(v00, v01)$, $(v01, v11)$, $(v11, v10)$, $(v10, v00)$). $S(v)$ is a 2-cube embedded in H_d. Let v and w be two distinct vertices of H, and p a path having v and w as its endpoints. We will define $C(p)$ as a configuration containing certain vertices and edges of H_d. The vertices which it contains are the eight vertices of $S(v) \cup S(w)$. The edges which it contains are the edges of the 2-cubes $S(v)$ and $S(w)$, together with the edges of the four paths $\phi_0(p), \ldots, \phi_3(p)$. $C(p)$ forms a *virtual 3-cube* in H_d. (See Fig. 3.) Note that the virtual cube $C(p)$ has the (somewhat disagreeable) property of sometimes containing an edge without containing the two endpoints of the edge, for the only vertices of p contained in $C(p)$ are the two endpoints of p. The four paths $\phi_0(p), \ldots, \phi_3(p)$ are edge disjoint, since they are contained in the disjoint $(d-2)$-cubes $\phi_i(H)$ in H_d $(0 \le i \le 3)$.

For distinct vertices v and w in H, $S(v)$ and $S(w)$ are vertex-disjoint and edge-disjoint configurations in H_d. If u, v, w are distinct vertices and p a path from v to w in H, then $S(u)$ and $C(p)$ are vertex disjoint; they are also edge disjoint, since their external edges are those in $S(u)$, $S(v)$, and $S(w)$, and $S(u)$ has no internal edges. Similarly, if v_1, w_1, v_2, w_2 are distinct vertices of H and p_1 and p_2 are edge-disjoint paths, with p_i having v_i and w_i as endpoints, then the virtual cubes $C(p_1)$ and $C(p_2)$ are both vertex disjoint and edge disjoint.

Proposition 3. *Let π_2 be a class $(a, b, c, 0)$ involution of the $(d-2)$-cube H. Let π_d be a class $(a, b, c+2, 0)$ involution of the d-cube H_d. If π is routable in H then π_d is routable.*

Proof. By the comments in Section 2, we may assume that π_d is the "extension" of π by tacking on paired coordinates $d-1$ and d: we may assume that $\pi_d(x_1x_2 \ldots x_d) = y_1y_2 \ldots y_d$ where $y_1 \ldots y_{d-2} = \pi_2(x_1 \ldots x_{d-2})$, $y_{d-1} = x_d$, and $y_d = x_{d-1}$. Let embeddings ϕ_0, ϕ_1, ϕ_2, ϕ_3 of H into H_d be defined as above.

Let P be a routing of π in H. Let $\mathscr{S} = \{S(v)\colon v \in H, v \text{ is fixed by } \pi\}$, and $\mathscr{C} = \{C(p)\colon p \in P\}$. Then $\mathscr{S} \cup \mathscr{C}$ is a collection of configurations which are both vertex disjoint and edge disjoint.

π_d fixes $H00$ and $H11$ as sets. (They are likely not pointwise fixed.) π_d maps vertices of $H01$ to $H10$, and vertices of $H10$ to $H01$. Also, π_d fixes (as a set) each $S(v)$ in $\mathscr{S}$ (since $S(v) \in \mathscr{S}$ implies that v is fixed by π), and each $C(p)$ in $\mathscr{C}$.

For each $S \in \mathscr{S}$, S is a 2-cube; the restriction of π_d to the vertices of S (denoted by $\pi_d|S$) is the same as the permutation in Example 1 induced by α. From Example 1 we see that there is a routing of $\pi_d|S$, which will be denoted by $P(S)$, and having the property that all its edges are contained in S. ($P(S)$ contains a single path; we may let that path be $(v01, v00, v10)$, where $S = S(v)$.)

Let $P(\mathscr{S}) = \bigcup\{P(S)\colon S \in \mathscr{S}\}$. $P(\mathscr{S})$ is a collection of edge-disjoint paths in H_d, because (1) for each $S \in \mathscr{S}$, every edge in $P(S)$ is contained in S, and (2) members of $\mathscr{S}$ are edge disjoint.

For each $C \in \mathscr{C}$, C is a virtual 3-cube in H_d. The permutation on vertices of C induced by π_d is the same as the permutation on the vertices of the 3-cube in Example 2 induced by β. From Example 2 we see that there is a routing of $\pi_d|C$, which will be denoted $P(C)$, and having the property that all its edges are contained in C.

Let $P(\mathscr{C}) = \bigcup\{P(C)\colon C \in \mathscr{C}\}$. $P(\mathscr{C})$ is a collection of edge-disjoint paths in H_d. Also, for each $C \in \mathscr{C}$, and $S \in \mathscr{S}$, $P(C)$ and $P(S)$ are edge disjoint.

Finally, $P^* = P(\mathscr{S}) \cup P(\mathscr{C})$ has the properties that (1) for every vertex v of H_d which is not fixed by π_d, some path in P^* has v and $\pi_d(v)$ as endpoints, and (2) paths in P^* are edge disjoint. Hence P^* is a routing of π_d. □

Proposition 4. *Let π be a class $(a, b, c, 0)$ involution of the $(d-2)$-cube H, and P be a routing of π in H. Let π_d be a class $(a, b+2, c, 0)$ involution of the d-cube H_d.*

(a) *If $b > 0$ then π_d is routable.*

(b) *Let μ be a pairing of the vertices of H which are fixed by π, so that no vertex is fixed by both π and μ. (We may consider μ to be a bijection on H, by saying that μ fixes each vertex of H which is not fixed by π.) Suppose that Q is a routing of μ in H, such that no edge is used in both P and Q. Then H_d is routable.*

It is natural to try to use the method of the proof of Proposition 3 prove this proposition. That method does not quite work if π fixes a vertex of H. The difficulty is that for a vertex v in H which is fixed by π, the action by π_d on the 2-cube $S(v) = S$ is a class $(0, 2, 0, 0)$ involution (which is antipodal); by Proposition 1 there is no routing of $\pi_d|S$ having the property that all edges are in S. The purpose of the pairing μ and paths in Q is to get us around this difficulty, by embedding S in a (virtual) 3-cube on which π_d acts as a class $(1, 2, 0, 0)$ automorphism.

Proof. We prove part (b) first; then part (a) is a simple corollary (or may be proven independently, much like Proposition 3). By the comments in Section 2, we may assume that π_d is defined by: $\pi_d(x_1x_2 \ldots x_d) = y_1y_2 \ldots y_d$ where $y_1 \ldots y_{d-2} = \pi(x_1 \ldots x_{d-2})$, $y_{d-2} = \bar{x}_{d-2}$, and $y_d = \bar{x}_d$.

Let $\mathscr{C} = \{C(p)\colon p \in P\}$, and $\mathscr{D} = \{C(q)\colon q \in Q\}$. As in the proof of Proposition 3, $\mathscr{C} \cup \mathscr{D}$ is a collection of configurations which are both vertex disjoint and edge disjoint. For each C in $\mathscr{C} \cup \mathscr{D}$, π_d fixes the set of vertices of C.

For each virtual 3-cube $C \in \mathscr{C}$, the permutation on vertices of C induced by π_d is the same as the permutation on the vertices of the 3-cube induced by the involution γ of Example 3. From Example 3 we see that there is a routing of $\pi_d|C$ in H_d, which will be denoted $P(C)$, and having the property that all its edges are contained in C.

For each virtual 3-cube $D \in \mathscr{D}$, the permutation on vertices of D induced by π_d is the same as the permutation on the vertices of the 3-cube in Example 4 induced by δ. From Example 4 we see that there is a routing of $\pi_d|D$ in H_d, which will be denoted $P(D)$, and having the property that all its edges are contained in D.

Let $P(\mathscr{C}) = \bigcup\{P(C)\colon C \in \mathscr{C}\}$ and $P(\mathscr{D}) = \bigcup\{P(D)\colon D \in \mathscr{D}\}$. $P(\mathscr{C})$ and $P(\mathscr{D})$ are collections of edge-disjoint paths in H_d. Also, for each $C \in \mathscr{C}$ and $D \in \mathscr{D}$, $P(C)$ and $P(D)$ are edge disjoint. $P(\mathscr{C}) \cup P(\mathscr{D})$ is a routing for π_d in H_d. This completes the proof of part (b).

Part (a) follows easily from part (b): since $b > 0$, no vertex is fixed by π. Then the empty collection of paths satisfies the properties demanded for Q in part (b). Hence π_d is routable. □

Example 6. We demonstrate that a class $(0, 2, 4, 0)$ involution is routable. To do this, we will apply Proposition 4(b) to an involution of class $(0, 0, 4, 0)$. A routing P for the class $(0, 0, 4, 0)$ involution π_4 which exchanges the first two coordinates, and exchanges the last two coordinates, is displayed in Example 5. To use Proposition 4(b) on π_4 we need additional paths, joining fixed points of π_4. The fixed points of π_4 are 0000, 1100, 0011, and 1111. A routing for a pairing of the fixed points is $Q = \{p_1, p_2\}$, where $p_1 = (0000, 0001, 0101, 0111, 0011)$ and $p_2 = (1111, 1110, 0110, 0100, 1100)$. Q is edge disjoint from P. Then by Proposition 4(b), a class $(0, 2, 4, 0)$ involution is routable. (Note: If we had generated the routing for π_4 by using Proposition 3 on a class $(0, 0, 2, 0)$ involution, we could not have then constructed Q.)

Proposition 5. *Let π be a class $(a, b, c, 0)$ involution of the $(d-1)$-cube H. Let π_d be a class $(a+1, b, c, 0)$ involution of the d-cube H_d. If π is routable then π_d is routable.*

Proof. Proof of this is fairly trivial. We may assume that π_d is defined by $\pi_d(x_1x_2 \ldots x_d) = y_1y_2 \ldots y_d$ where $y_1y_2 \ldots y_{d-1} = \pi(x_1x_2 \ldots x_{d-1})$ and $y_d = x_d$. H_d contains two disjoint copies of H, namely $H0$ and $H1$, and π_d fixes (as a set) each of $H0$ and $H1$. Let ϕ_0 be the map from H to $H0$ which maps each vertes v to $v0$, and ϕ_1

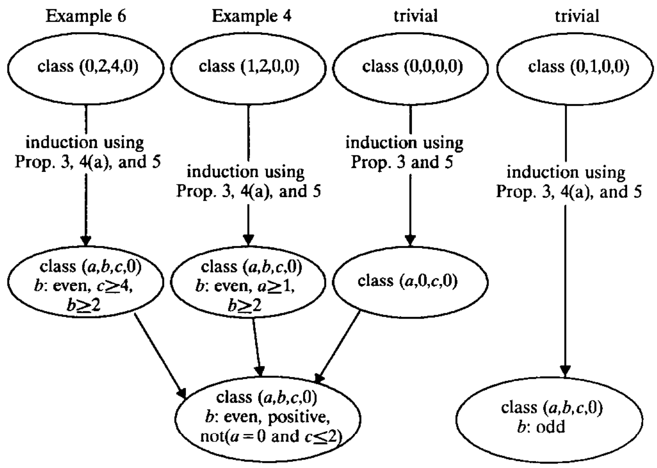

Fig. 4. Proof of existence of routings for class $(a, b, c, 0)$ involutions, where a, b, c do not satisfy: $a = 0, c \leq 2$, and b even and positive.

be the corresponding map from H to $H1$. Then, where P is a routing of π, $\phi_0(P) \cup \phi_1(P)$ is a routing of π_d. $\square$

Proof of the Theorem. Let π be an involution of H_d. We are to show that if d is odd, π is routable; if d is even, π is routable iff π is not of class $(0, d, 0, 0)$ (where $d \geq 2$) or $(0, d-2, 2, 0)$ (where $d \geq 4$). Equivalently, we are to show that a class $(a, b, c, 0)$ involution of a hypercube fails to be routable iff $a = 0$, $c \leq 2$, and b is even and positive. (Note that for every involution, c is even.) Figure 4 illustrates the proof of the positive part of the assertion.

We split this proof into three cases, according as b is odd, zero, or even and positive. We handle an easy case, where b is odd, first. The smallest such involution is the class $(0, 1, 0, 0)$ involution on H_1. It is trivially routable. Using Proposition 5, and then Proposition 3, we see that all involutions of class $(a, 1, 0, 0)$, and then $(a, 1, c, 0)$ (where $a \geq 0, c \geq 0$) are routable. Finally, by Proposition 4(a) all involutions of class $(a, b, c, 0)$ where b is odd are routable.

The case where $b = 0$ is similar. The class $(0, 0, 0, 0)$ involution on H_0 is routable. By Propositions 3 and 5 all involutions of class $(a, 0, c, 0)$ (where $a \geq 0$, $c \geq 0$) are routable.

To start the case where b is even and positive, we note that class $(1, 2, 0, 0)$ and $(0, 2, 4, 0)$ involutions are routable, by Examples 4 and 6. Propositions 3 and 5 may be used to construct routings for all involutions of class $(a, 2, c, 0)$, excluding those of class $(0, 2, 0, 0)$ and $(0, 2, 2, 0)$. Finally, for b positive and even, Proposition 4(a) may be used to construct routings for all involutions of class $(a, b, c, 0)$, excluding classes $(0, b, 0, 0)$ and $(0, b, 2, 0)$. Involutions of classes $(0, b, 0, 0)$ and $(0, b, 2, 0)$ are not routable, by Propositions 1 and 2. $\square$

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